PARTITIONING GRAPHS SUBJECT TO EDGE CONSTRAINTS

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THESIS

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ABSTRACT

A problem of partitioning a given graph into a minimal number of subgraphs subject to edge and node constraints is considered. Two parameters associated with the subgraph, one corresponding to the maximum number of nodes and the other to the maximum number of external edges, define a feasible partition element.

Complete graphs, complete bipartite graphs, and two families of infinite graphs are considered, and relations between the parameters are used to obtain the results. For the infinite graphs, the problem is somewhat different. A largest feasible partition element is found and can be used in determining the minimal number of feasible elements in a finite graph with the same structure as the infinite one.

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I. INTRODUCTION

The field of computer design has produced many problems which can be solved by means of graph theory or combinatorial analysis. One such problem is the partition problem as defined by Kodres [1], where two numbers are associated with a circuit element corresponding to the number of external connections and the amount of space the element requires.

The problem is taken here and defined with the nodes of a graph corresponding to the space requirement and particular edges corresponding to the external connections. The graph is then partitioned so that these two quantities do not exceed specified limits, and the problem is to minimize the number of partition elements.

Other authors have studied variations of this particular problem. Luccio and Sami [2] have considered the problem of decomposing a network into a number of subnetworks such that the number of interconnections among them is minimal under specified conditions. They have presented and proved a number of different properties of minimal groups and based on these, they have developed a procedure to determine all the minimal groups of a given network.

Lawler, Levitt, and Turner [3] have considered the problem of determining a partitioning that will result in a network where maximum delay is minimized throughout the

network. They have assumed that delays occur in external connections, whereas no delays occur in the internal connections.

After the basic terms are defined and the problem is specifically formulated in Section II, complete graphs are studied in Sections III and IV. Complete bipartite graphs are taken up in Section V. Two cases of infinite graphs are studied in Sections VI and VII, both being regular of degree four, but one will be in two dimensions and the other in one dimension.

II. DEFINITIONS AND FORMULATION OF THE PROBLEM

In order to give precise meaning to some fundamental concepts, we first introduce the idea of an unordered product of a set with itself as developed by Busacker and Saaty [4]. The symbol (s & t) will note the unordered product of a pair of elements from some set S, and the collection of all such pairs will be denoted by S & S. This will be called the unordered product of a set S with itself. Hence, if $s \varepsilon S$ and $t \varepsilon S$, the symbols (s & t) and (t & s) denote the same thing.

We can now define a graph as follows: A graph consists of a nonempty set V, a set E disjoint from V, and a mapping Φ of E into V & V. The elements of V are called nodes; the elements of E are called edges; and the mapping Φ is called the incidence mapping associated with the graph. The graph will be denoted by G(V,E) or simply G.

If $e \varepsilon E$ such that $\Phi(e) = (v \& w)$ where $v \varepsilon V$ and $w \varepsilon V$, then the edge e is said to be incident with each of the nodes, and v and w are called the end points of e. If $\Phi(e) = (v \& w)$ where $v = w$, then e is called a loop. If $\Phi(e_1) = (v \& w)$ and $\Phi(e_2) = (v \& w)$, then e_1 and e_2 are called parallel edges. We will be concerned here only with those graphs that have no loops and no parallel edges.

A finite sequence of not necessarily distinct edges such that one end point of the first edge is also an end point of the second, the remaining end point of the second

is also an end point of the third, etc., is called an edge progression. The edge progression is said to be closed if the "free" end point of the first edge is the same node as the "free" end point of the last edge. A circuit progression is a closed edge progression having no repeated edges. A circuit is a set of edges which, if properly ordered, forms a circuit progression. The order of a circuit is defined to be the number of edges in a circuit.

With these ideas in mind, we can now move toward defining the partitioning problem.

Definition 1. If G is a graph with a set of nodes V, then a family of nonempty subsets of V, $\{V_i\}$ $i \in I$, is called a partition of G if:

i) $\bigcup_{i \in I} V_i = V$,

ii) $V_i \cap V_j = \phi$ for $i \neq j$.

Any such subset V_i will be called a partition element. The indexing set I may be finite or infinite.

Definition 2. If V_i is a subset of V and e an edge with end points v and w, then e is said to be an external edge if $v \in V_i$ and $w \in V - V_i$. If both $v \in V_i$ and $w \in V_i$, then e is said to be an internal or buried edge.

An external edge is said to be an external edge of a partition element if one of its end points is a member of that particular partition element. Thus, the term "number of external edges of a partition element" takes on meaning and gives rise to the following definition.

Definition 3. A partition element is said to be edge feasible if it has a number of external edges which is less than or equal to some given integer s. The integer s is called the edge feasibility constant.

Definition 4. A partition element is said to be node feasible if it contains a number of nodes which is less than or equal to some given integer z. The integer z is called the node feasibility constant.

Definition 5. A partition element is said to be feasible if it is both node feasible and edge feasible. A partition of G is said to be feasible if every partition element is feasible.

The general problem can now be stated for both finite and infinite graphs. For finite graphs, the problem is to find a partitioning with the minimum number of feasible partition elements. For infinite graphs, the problem is to determine the largest feasible partition elements. In general, both the finite and infinite partitioning problems present great difficulties because the structure of a graph with N nodes and M edges determines the number of partition elements. If the structure is regular, then there is a possibility of finding some relationship between the number of nodes and the number of external edges in a partition element. In these cases, some concise method for solving the problem may exist. The next sections will present some of these cases.

Before going on, a remark is needed to clarify some of the methods used. Since we will use only integer values, equations will be solved and answers changed to integers using the following symbols: $[x]$ will be used to represent "the greatest integer less than or equal to x," and $\{x\}$ will be used to represent "the smallest integer greater than or equal to x."

III. COMPLETE GRAPHS

In this section complete graphs will be studied since they offer a regular structure and a straight-forward approach to a solution. We begin by defining complete graphs as follows: A graph $G_N(V,E)$ is a complete graph of N nodes if every pair of distinct nodes is connected by an edge. Therefore, let G_N be a complete graph of N nodes. Let $\{V_i\}$ $i = 1,2,...K$ be a feasible partition of G_N, and let s and z be the two constants of feasibility. The problem in this case is to determine the minimum number of partition elements, that is, a minimum K.

One method of approach is to find a partition in which every partition element is node feasible and then check if it is also edge feasible. If it is, then the problem is essentially solved. If it is not, then some other node feasible partition must be found. Of course, not every feasible partition will be a solution, since we are looking for the one with a minimum number of partition elements.

To establish some motivation for choosing a particular partitioning, notice that if a partition element in a complete graph contains x nodes, then it has x(N-x) external edges. This is an increasing function of x up to $x = \frac{1}{2}N$. Since the number of external edges of a partition element must be kept under a specified limit, the number of nodes in a partition element should be as large as possible. But

this also implies that the number of buried edges is as large as possible, since if a partition element has x nodes, it has $\frac{1}{2}x(x-1)$ buried edges, which is also an increasing function of x. This observation leads to the following definition.

Definition 6. An optimal partitioning of G is one in which every partition element is node feasible and $\sum_{i=1}^{K} \gamma(V_i)$ is maximum, where $\gamma(V_i)$ is the number of buried edges in the partition element V_i.

Lemma 1. Let $\{V_i\}$ $i = 1,2,\ldots,K$ be a partition of a complete graph G_N, where every partition element is node feasible. If z is the given node feasibility constant, then an optimal partitioning of G_N is one in which every partition element contains z nodes except possibly one.

Proof: Assume there exists at least two partition elements which do not contain z nodes. Let one of them contain f nodes and the other g nodes where $f < g < z$. If this partitioning is optimal, then the number of buried edges in the partition element with f nodes plus the number of buried edges in the partition element with g nodes is not less than the number of buried edges in some conceivable partition elements with z and r nodes, where $z = g + y$ and $r = f - y$, with $y > 0$. Thus, $\frac{1}{2}f(f-1) + \frac{1}{2}g(g-1) \geq \frac{1}{2}z(z-1) + \frac{1}{2}r(r-1)$. After substituting for z and r and solving, the following is obtained: $y^2 + gy - fy \leq 0$. Since $y > 0$, $y + g - f \leq 0$.

Substituting for y gives $(z-g)+g-f \leq 0$, or $z-f \leq 0$. But $z-f > 0$ by assumption; thus, a contradiction arises and we have as a conclusion $N = p \cdot z + r$, where $0 \leq r \leq z-1$, and p is some integer.

We now have obtained a way to partition G_N such that every partition element is node feasible. The following theorem provides a way of checking the edge feasibility criterion and thus, solves the problem.

Theorem 1. Let G_N be a complete graph of N nodes, and let z and s be the node feasibility and the edge feasibility constants respectively. The minimal number of feasible partition elements $\{V_i\}$ $i = 1,2,\ldots,K$ is given by

$$K = \max(\{N/z\}, \{N/w\}) \text{ where}$$

$$w = [1/2(N - \sqrt{N^2 - 4s})].$$

Proof: Let $\{V_i\}$ $i = 1,2,\ldots,K$ be an optimal partitioning of G_N. Then $N = p \cdot z + r$, where $0 \leq r \leq z-1$ and p is some integer. If every partition element is edge feasible, then $z(N-z) \leq s$ and $r(N-r) \leq s$. If $r = 0$, then $N = p \cdot z$, hence $p = N/z$. If $r \neq 0$, then $p = [N/z]$. In the first case $K = N/z$; in the second case $K = [N/z]+1$ or $K = \{N/z\}$.

If the node feasible partition elements are not edge feasible, then let w be an integer such that $w < z$, and w is the largest integer such that $w(N-w) \leq s$. Solving the inequality for w gives $w \leq \frac{1}{2}N - \frac{1}{2}\sqrt{N^2-4s}$. Since w is to be the largest integer for which $w(N-w) \leq s$,

$w = [\frac{1}{2}N - \frac{1}{2}\sqrt{N^2-4s}]$. If w is now considered as a new node feasibility constant, and G is partitioned optimally, then $N = q \cdot w + r_1$, where $0 \leq r_1 \leq w-1$, and q is some integer. From the first part of the proof, $K = \{N/w\}$.

As an example, suppose G is a complete graph of 462 nodes and let z = 32 and s = 10000. Then $\{N/z\} = \{462/32\} = \{14 \frac{14}{32}\} = 15$. $\{N/w\} = 462/[\frac{1}{2}(462) - \frac{1}{2}\sqrt{(462)^2 - 4(10000)}] = \{21\} = 21$. Thus K = 21.

IV. LARGEST COMPLETE GRAPH

Turning aside from the main theme of the last section, we note a related problem which is stated as follows: Let G_N be a complete graph of N nodes. Let z and s be the node feasibility constant and edge feasibility constant respectively. If G_N is partitioned optimally, what is the maximum number of nodes N that G_N may contain such that the edge feasibility criterion is satisfied?

If G_N is partitioned optimally, then $N = p \cdot z + r$, where $0 \leq r \leq z-1$, and p is some integer. Since G_N is complete, the total number of edges in G_N is $\frac{1}{2}N(N-1)$. The total number buried edges is $\frac{1}{2}pz(z-1) + \frac{1}{2}r(r-1)$, and the total number of external edges is $\frac{1}{2}(ps + r(N-r))$ if the number of external edges of a partition element is s when it contains z nodes. Thus, $\frac{1}{2}N(N-1) = \frac{1}{2}pz(z-1) + \frac{1}{2}r(r-1) + \frac{1}{2}ps + \frac{1}{2}r(N-r)$. Thus $\frac{1}{2}N(N-1) = \frac{1}{2}pz(z-1) + \frac{1}{2}r(r-1) + \frac{1}{2}ps + \frac{1}{2}r(N-r)$. Substituting $(N-pz)$ for r and solving for N gives $N = z + s/z$. Since N must be an integer, $N = [z + s/z]$.

V. COMPLETE BIPARTITE GRAPHS

We now return to another family of graphs and consider the general problem. Even though an algorithm will be given to obtain a solution, there will be no proof that the solution obtained will have the minimum number of feasible partition elements. The problem here is that some assumptions will be made on the construction of an optimal partitioning, and although some arguments will be presented to substantiate the assumptions, these arguments will not constitute a proof.

We begin by defining the class of graphs which will be studied. A graph is said to be bipartite if its nodes can be partitioned into two disjoint sets A_1 and A_2 such that every edge has one end point in A_1 and the other end point in A_2. A bipartite graph is said to be a complete bipartite graph if every node in A_1 is connected to every node in A_2. Let $G_{M,N}(V,E)$ be a complete bipartite graph with M nodes in one section and N nodes in the other with $M \leq N$. Let $\{V_i\}$ $i = 1,2,\ldots,K$ be a feasible partitioning of $G_{M,N}$ with given feasibility constants z and s. The problem, once again, is to obtain the feasible partitioning with the minimum number of feasible partition elements.

We assume here that if $G_{M,N}$ is partitioned optimally, then $M + N = q \cdot z + r$ where $0 \leq r \leq z-1$ and q is some integer. Consider now a partition element with z nodes, and divide

it into two sections, one with t nodes and one with (z-t) nodes. Assume now that $t/M = (z-t)/N$. This gives a value of t as $t = \frac{z \cdot M}{M+N}$. Since this quotient is not in general an integer, we assume the partition is constructed as follows:

$$M = \alpha_1 [t] + \alpha_2 \{t\} + r_1$$

$$N = \alpha_1 (z-[t]) + \alpha_2 (z-\{t\}) + r_2$$

where $0 \leq r_1 + r_2 \leq z-1$ and α_1, α_2 are integers.

To give an argument to justify that this partitioning is optimal, first observe that adding the two expressions will give $M+N = q \cdot z + r$, which is the assumed structure of an optimal partitioning. Notice, too, that a function giving the number of external edges for a partition element is $t(N-(z-t)) + (z-t)(M-t)$. (We are working here with a real value of t, but once a solution is found, we will change the result to an integer value.) Let z be fixed for the moment and call this function f(t). Collecting terms will give $f(t) = 2t^2 - (2z+M-N)t + Mz$. If $df/dt = 0$, then $t = \frac{1}{4}(2z+M-N)$. Let $t_o = \max(\frac{1}{4}(2z+M-N), \frac{1}{2})$ and let

$$M = \beta_1 [t_o] + \beta_2 \{t_o\}$$

$$N = \beta_1 (z - [t_o]) + \beta_2 (z - \{t_o\}),$$

where β_1 and β_2 need not be integers.

It is required that both β_1 and β_2 be nonnegative. If solving for β_1 and β_2 in the above system gives at least one of them a negative value, we let $t_1 = t_o + 1$ and

solve $\quad M = \beta_1[t_1] + \beta_2\{t_1\}$

$$N = \beta_1(z - [t_1]) + \beta_2(z - \{t_1\}).$$

This process is repeated until both coefficients are non-negative, say on the i^{th} step. This allows the following observation to be made:

$$[t_i] = \left[\frac{z \cdot M}{M+N}\right], \quad [t_i] = \left\{\frac{z \cdot M}{M+N}\right\}.$$

As an example, suppose $G_{10,25}$ is given with a node feasibility constant of 6. Solving for $t = \frac{1}{4}(2z + M - N)$ gives $t = -3/4$ and $t_o = \max\ (-3/4,\ 1/2) = 1/2$.

$$10 = \beta_1 \cdot 0 + \beta_2 \cdot 1$$

$$25 = \beta_1 \cdot 6 + \beta_2 \cdot 5$$

Solving gives $\beta_1 = -25/6$, $\beta_2 = 10$. Let $t_1 = \frac{1}{2} + 1$.

$$10 = \beta_1 \cdot 1 + \beta_2 \cdot 2$$

$$25 = \beta_1 \cdot 5 + \beta_2 \cdot 4$$

This gives $\beta_1 = 5/3$, $\beta_2 = 25/6$. Notice that

$t = \frac{zM}{M+N} = 6 \cdot 10/35 = 1\ 5/7$, and

$$[t_1] = [1\ 5/7] = 1 \quad \text{and} \quad \{t_1\} = \{1\ 5/7\} = 2.$$

Looking at the partitioning problem from another point of view gives the following argument. Since we are trying to insure optimality, we would like to find the node feasible partition which gives the largest number of buried edges. In a partition element of z nodes, the number of buried edges is $t(z-t)$. This is maximum at $t = z/2$. Hence let

$$M = \gamma_1 \cdot z/2 + r_1$$

$$N = \gamma_1 \cdot z/2 + (z-r_1) + c_1 \cdot z + r_1'.$$

If $\frac{1}{2}z$ is not an integer, use

$$M = \gamma_1 \cdot [z/2] + r_1$$

$$N = \gamma_1 \cdot \{z/2\} + (z-r_1) + c_1 \cdot z + r_1'.$$

Calculate the number of internal edges. If $t_o = [z/2]$, let $t_1 = [z/2] - 1$ and consider

$$M = \gamma_2[t_1] + r_2$$

$$N = \gamma_2(z - [t_1]) + (z-r_2) + c_2 z + r_2'.$$

Calculate the number of internal edges. Repeat these steps until the number of internal edges is maximum, and again notice that it occurs when $[t_i] = \left[\frac{zM}{M+N}\right]$.

Looking at the example of $G_{10,25}$ $z = 6$ again. Let $t_o = [z/2] = 3$. Then

$$10 = 3 \cdot 3 + 1$$

$$25 = 3 \cdot 3 + 1 \cdot (5) + 1 \cdot 6 + 1 \cdot 5$$

The total number of buried edges is

$$3(3 \cdot 3) + 1 \cdot (1.5) + 0 + 0 = 32$$

Now let $t_1 = 2$. Then

$$10 = 5 \cdot 2 + 0$$

$$25 = 5 \cdot 4 + 0 + 0 + 5$$

The total number of buried edges is

$$5(2 \cdot 4) + 0 + 0 + 0 = 40.$$

The next step has to be modified since

if $\quad 10 = (10 \cdot 1)$ then $\gamma = 10$, so that

$$25 = 10 \cdot (6-1) > 25.$$

So we let $10 = 1 \cdot 1 + 4 \cdot 2 + 1$

$$25 = 1 \cdot 5 + 4 \cdot 4 + 4$$

which gives 41 buried edges. Notice that

$$1 = \left[\frac{zM}{M+N}\right] \quad \text{and} \quad 2 = \left\{\frac{zM}{M+N}\right\}.$$

With these heuristic arguments in mind, we can now assert that an optimal partition of $G_{M,N}$ is given by

$$M = \alpha_1 [t] + \alpha_2 \{t\} + r_1$$

$$N = \alpha_1 (z-[t]) + \alpha_2 (z-\{t\}) + r_2 \quad \text{where}$$

$t = zM/M+N$, $0 \leq r_1 + r_2 \leq z-1$, and α_1, α_2 are integers.

To find the integers α_1 and α_2, first let

$t = \left[\frac{Mz}{M+N}\right] + f$, $\quad f = \frac{Mz}{M+N} - \left[\frac{Mz}{M+N}\right]$ and solve

$$M = \alpha_1 (t-f) + \alpha_2 (t+1-f)$$

$$N = \alpha_1 (v-t+f) + \alpha_2 (v-t-1+f) .$$

This gives $\alpha_1 = (1-f)(\frac{M+N}{z})$, $\alpha_2 = f(\frac{M+N}{z})$. Since α_1 and α_2 to be integers, we find them by the following rules. If α_1 and α_2 are both less than an integer plus one half (one of them may equal one half), then use $[\alpha_1]$ and $[\alpha_2]$. If one of them is less than an integer plus one half and the other is greater than an integer plus one half, then the smallest is reduced to the greatest integer less than or equal to it, and the larger is increased to the smallest integer greater

than or equal to it. This rule holds provided the resulting sum is less than or equal to $\frac{M+N}{z}$. If the sum is greater than $\frac{M+N}{z}$ then both α_1 and α_2 are reduced to $[\alpha_1]$ and $[\alpha_2]$. If both are greater than an integer plus one half, then α_1 becomes $[\alpha_1]$ and α_2 becomes $\{\alpha_2\}$.

For example, in the previous case $t = 1\ 5/7$, $f = 5/7$, $\alpha_1 = 1\ 2/3$, $\alpha_2 = 4\ 1/6$. Since $\frac{M+N}{z} = 5\ 5/6$, $\alpha_1 + \alpha_2 \leq 5\ 5/6$. Hence if α_1 were increased to 2 and α_2 reduced to 4, this would give a value greater than 5 5/6. Therefore $\alpha_1 = 1$, $\alpha_2 = 4$, and the remainders r_1 and r_2 pick up the remaining nodes.

To solve the problem of obtaining a minimal number of feasible partition elements K, we must consider the optimal partitioning structure of $G_{M,N}$. Let $M = \alpha_1[t] + \alpha_2\{t\} + r_1$

$$N = \alpha_1(z-[t]) + \alpha_2(z-\{t\}) + r_2,$$

where α_1 and α_2 are integers and $0 \leq r_1 + r_2 \leq 2 - 1$. As previously obtained, the number of external edges is given by $f(t) = 2t^2 - (2z+M-N)t + zM$. Since $f([t]) \leq f(\{t\})$ for $0 \leq t \leq z/2$, we need only consider $f(\{t\})$ to check edge feasibility as long as α_2 is greater than zero. If α_2 equals zero, then the procedure is the same except $f([t])$ is used instead of $f(\{t\})$.

If $f(\{t\}) \leq s$, $K = \left\{\frac{M+N}{z}\right\}$. If $f(\{t\}) > s$, then let $z_1 = z-1$. Solve for a new optimal partitioning structure of $G_{M,N}$. This will give $t_1 = \frac{Mz_1}{M+N}$ and

$M = \alpha_1[t_1] + \alpha_2\{t_1\} + r_1'$

$N = \alpha_1(z-[t_1]) + \alpha_2(z-\{t_1\}) + r_2'$ where the α's may be different. If $f(\{t_1\}) \leq s$, then

$K = \left\{\frac{M+N}{z_1}\right\}$. If $f(\{t_1\}) > s$, let

$z_2 = z_1 - 1$ and go through the procedure again. This process is continued until the edge feasibility criterion is satisfied.

VI. REGULAR GRAPHS OF DEGREE FOUR IN TWO DIMENSIONS

The graphs in this section will be infinite, and therefore the problem will be to determine the largest feasible partition element. Let G(V,E) be an infinite regular graph of degree four as pictured in Figure 1.

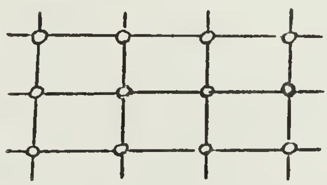

Figure 1. An infinite regular graph of degree four.

Let a subset of V contain x nodes. It is required to find the structure of such a subset so that the number of buried edges is maximum. Because of the structure of G, a partition element of x nodes will contain more circuits if the nodes are "close together" instead of "spread out." For example, if x = 13, then the graph in Figure 2 contains more circuits than that in Figure 3.

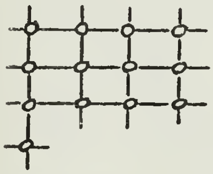

Figure 2. A partition element of 13 nodes which are close together."

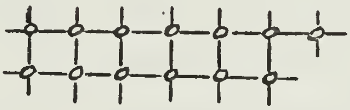

Figure 3. A partition element which is "spread out."

Since it contains more circuits, it contains more buried edges.

Therefore, to obtain a largest feasible partition element given it is edge feasible with given constant s, we must first find a function of z which gives a value of s using the above construction.

Lemma 2. If the value of z is a perfect square, then the value of s is given by $s = 4\sqrt{z}$.

Proof. If z is a perfect square, then the partition element has $\sqrt{z}$ rows containing $\sqrt{z}$ nodes, because of the optimal structure assumed for a partition element. This implies that the number of external edges on one side of the element is $\sqrt{z}$. Since there are four sides, $s = 4\sqrt{z}$.

Lemma 3. If $[\sqrt{z}]^2 < z < \frac{1}{2}([\sqrt{z}]^2 + ([\sqrt{z}]+1)^2)$, then the value of s is given by $s = 4[\sqrt{z}] + 2$.

Proof: If a perfect square z has $4\sqrt{z}$ external edges, then adding one node puts z within the hypothesis of this lemma. One of the external edges becomes a buried edge and three new external edges are added. $s = 4[\sqrt{z}] - 1 + 3 = 4[\sqrt{z}] + 2$. If another node is added and the hypothesis is still satisfied, then two external edges become internal edges and two new internal edges are added. $s = 4[\sqrt{z}] + 2 - 2 + 2 = 4[\sqrt{z}] + 2$. This is repeated until $z = \frac{1}{2}([\sqrt{z}]^2 + ([\sqrt{z}] + 1)^2) - \frac{1}{2}$, which will give $[\sqrt{z}]$ rows of $[\sqrt{z}] + 1$ nodes in each row.
$s = 2([\sqrt{z}]) + 2([\sqrt{z}] + 1) = 4[\sqrt{z}] + 2$.

Lemma 4. If $\frac{1}{2}([\sqrt{z}]^2 + ([\sqrt{z}] + 1)^2) < z < ([\sqrt{z}] + 1)^2$, then $s = 4[\sqrt{z}] + 4$.

Proof: Starting with the last result and adding a node gives a value of z within the hypothesis in this lemma. This gives three new external edges while making an internal edge out of one of the old external edges. $s = 4[\sqrt{z}] + 2 - 1 + 3 = 4[\sqrt{z}] + 4$. Adding another node makes two internal edges out of two external edges and adds two new external edges. $s = 4[\sqrt{z}] + 4 - 2 + 2 = 4[\sqrt{z}] + 4$. This is continued until z is again a perfect square and lemma 1 is applied.

Lemma 5. The value of s is always even.

Proof: From lemma 2, $s = 4[\sqrt{z}] = 2(2\sqrt{z})$.
From lemma 3, $s = 4[\sqrt{z}] + 2 = 2(2[\sqrt{z}] + 1)$.
From lemma 4, $s = 4[\sqrt{z}] + 4 = 2(2[\sqrt{z}] + 2)$.

Theorem 2. If s is the given edge feasibility constant, then the maximum number of nodes per partition is $z = \left[\frac{1}{16} s^2\right]$.

Proof: By lemma 5, the number of external edges is even for all z, thus if s is odd, reduce it to the next lower integer which is even. If $s \equiv 0 \bmod 4$, then the maximum number of nodes in a subset will be arranged in a perfect square structure with $\frac{1}{4}s$ external edges on a side. Thus, there are $\frac{1}{4}s$ rows of $\frac{1}{4}s$ nodes per row in the partition element. This implies $z = (\frac{1}{4}s)(\frac{1}{4}s) = 1/16s^2 = 1/16s^2$. If $s \equiv 2 \bmod 4$, then the maximum

number of nodes in a subset will be arranged in a rectangular structure with $\frac{1}{2}(\frac{1}{2}s-1)$ rows of $\frac{1}{2}(\frac{1}{2}s+1)$ nodes per row. This implies that

$$z = (\tfrac{1}{2})(\tfrac{1}{2}s-1)(\tfrac{1}{2})(\tfrac{1}{2}s+1) = 1/16 \cdot s^2 - \tfrac{1}{4} = \left\lfloor 1/16 \cdot s^2 \right\rfloor.$$

This largest partition element can now be used to solve a problem where a finite graph has an internal structure like the one of the infinite graph just studied. If z and s are the given feasibility constants, a largest partition element may be found. Of course, a better solution may be possible since the finite graph is different in its structure.

For example, let G be as shown in Figure 4, with given constants $z = 6$, $s = 8$. By Theorem 2, the maximum $z = 4$, and in this case $z = 4$ limits the size of the partition elements. Therefore, the minimum number of feasible partition elements is $K = \{20/4\} = 5$. This can be seen in Figure 4a. However, in Figure 4b with $K = 4$, the feasibility constants are still satisfied.

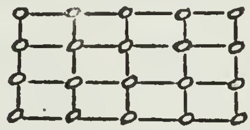

Figure 4. A finite graph of 20 nodes with internal structure similar to the infinite graph in Figure 1.

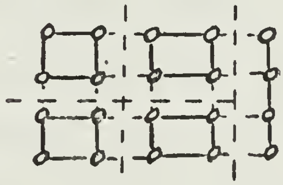

Figure 4a. The graph in Figure 4 partitioned with K = 5.

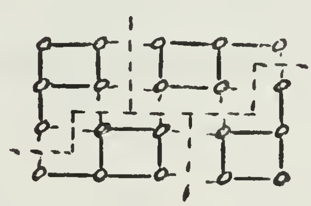

Figure 4b. The graph of Figure 4 partitioned with K = 4.

VII. REGULAR GRAPHS OF DEGREE FOUR IN ONE DIMENSION

Let G be a regular graph of degree four such that it has circuits of order three as pictured in Figure 5.

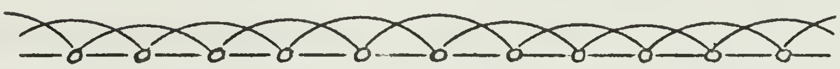

Figure 5. Regular graph of degree four and circuits of order three.

Here the largest value of s will be 6 no matter how many nodes are put in a partition element.

In general, let G be a regular graph of degree four with a structure as in Figure 5, but with circuits of minimal order c. The problem in this case is to determine the maximum value of s as a function of the circuit order. If a partition element is as large as possible such that it contains no circuits, it will have (c-1) nodes. The total number of edges associated with this partition element is 4(c-1) - ((c-1)-1), and the number of internal edges is (c-1)-1. This gives as the number of external edges 2c. If we add x nodes to this partition element, the total number of edges is 4(c-1+x) - ((c-1) - 1 + 3x), and the number of internal edges is ((c-1)-1+x), still giving 2c external edges. Thus, if the circuit order is c, then maximum s = 2c. If z, a node feasibility constant is given, then s = 2z + 2 up to z = c-1. After this point s is constant at 2c.

VIII. CONCLUSION

There has been no attempt here to reach conclusions or to form hypotheses concerning graphs in general. Although this study is limited to complete graphs, complete bipartite graphs, and two cases of infinite graphs, many other classes of graphs exist where the partitioning problem may be solved.

Some general statements can be made concerning these cases. If the node feasibility constant is large compared to the edge feasibility constant, then the minimum number of feasible partition elements will be a function of the edge feasibility constant. On the other hand, the node feasibility constant will determine the minimum number of partition elements if it is small compared to the edge feasibility constant. When comparing the two constants, the relation between them must be considered as a function of the structure of the particular graph.

If the edges of a graph are arranged such that they form a regular structure and are high in density compared to the nodes, then a greater number of them can be buried within the partition elements.

Other classes of graphs which are likely to have concise solutions are regular graphs of arbitrary degree, which have a specified structure such as the infinite graphs studied here.

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13. ABSTRACT

A problem of partitioning a given graph into a minimal number of subgraphs subject to edge and node constraints is considered. Two parameters associated with the subgraph, one corresponding to the maximum number of nodes and the other to the maximum number of external edges, define a feasible partition element.

Complete graphs, complete bipartite graphs, and two families of infinite graphs are considered, and relations between the parameters are used to obtain the results. For the infinite graphs, the problem is somewhat different. A largest feasible partition element is found and can be used in determining the minimal number of feasible elements in a finite graph with the same structure as the infinite one.

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